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SUBSPACES OF ASPLUND BANACH SPACES WITH THE POINT CONTINUITY PROPERTY

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ABSTRACT

We show that if X is a Banach space with the point continuity property and is Asplund then X is somewhat quasireflexive.

1. Introduction

A fundamental problem in infinite dimensional Banach space theory is: "Does every Banach space contain a subspace which is isomorphic to c_0, isomorphic to l^1, or reflexive?".

In this direction, Johnson and Rosenthal showed that if X^{**} is separable then X is somewhat reflexive [13]. Bellenot proved that this conclusion could be strengthened to: X is somewhat quasireflexive ("in a strong sense" [1]). A space is called *somewhat quasireflexive* provided every non-reflexive subspace has a subspace which is quasireflexive of order one. X has the *point continuity property* (PC) if every weakly closed bounded subset of X has a point of weak to norm continuity [17]. Edgar and Wheeler extended the result of Johnson and Rosenthal to the case when X has (PC) and is an Asplund space (X has property (PCA)) [6]. The class of the Banach spaces with (PCA) is larger than that of the Banach spaces with separable bidual: the predual of James Tree space JT has (PCA) but JT* is not separable [12], [16]; our main goal in this paper is to extend the result of Bellenot to Banach spaces with (PCA). More precisely, we prove (Theorem 1) that if X is a non-reflexive Banach space with separable dual and (PC) then each element of $X^{**} \setminus X$ is the weak* limit of a boundedly

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complete basic sequence (x_n) in X such that $[x_n]$ is quasireflexive of order one. Consequently, if X has (PCA) then X is somewhat quasireflexive (Corollary 3). In particular, this gives the results of Johnson and Rosenthal, Bellenot, Edgar and Wheeler.

This work was motivated by the following question asked by A. Pelczynski in Oberwolfach: "Does the space constructed by Bourgain and Delbaen, called Y (in [2], [3]), contain a quasireflexive subspace"? The proof of our main result (Theorem 1) uses the ideas of Ghoussoub–Maurey [10] and the construction of basic sequences [7], [8].

2. Notations, Definitions

For a Banach space X, we denote by $B(X)$ (resp. $S(X)$) the unit ball (resp. unit sphere) of X; we write X^* for the dual space; $[x_n]$ denotes the norm-closed linear span of the sequence (x_n). If $A, B \subset X$, $d(A, B) = \inf\{ \| x - y \| , x \in A, y \in B\}$. A sequence (x_n) is *basic* if and only if there exists a number $K > 0$ such that, for all integers p and q, with $p \leqq q$, and all scalars $a_1, \ldots, a_q$, one has:

$$\left\| \sum_{n=1}^{p} a_n x_n \right\| \leqq K \left\| \sum_{n=1}^{q} a_n x_n \right\| .$$

(x_n) is a *basis* of X if it is a basic sequence and if $[x_n] = X$. A basic sequence is called *boundedly complete* provided that $(\Sigma_{i=1}^{n} a_i x_i)_n$ is convergent whenever it is bounded.

A basic sequence (x_n) is called *shrinking* provided the functionals (x_n^*) biorthogonal to (x_n) form a basis for $[x_n]^*$. We refer the reader to [4], [18] for the study of basic sequences in Banach spaces.

A Banach space X is *quasireflexive* (*of order n*) if the codimension of X in its bidual is n.

A Banach space X is an *Asplund space* if every separable subspace of X has a separable dual space.

3. Property (PCA)

The results of the previous section produce structural information about spaces with property (PCA). Here are some examples of spaces with (PCA): spaces with separable second dual, the predual of the James Tree space [12], the James–Lindenstrauss spaces JL(S) modelled on separable Banach spaces S

[11], [15], a space constructed by Bourgain and Delbaen and called Y in [2], [3], the long James spaces $J(\omega_1)$ [5].

We recall some results concerning (PCA). Let X be a separable space; X has (PCA) if and only if $(B(X), \omega)$ is a Polish space [6].

If X contains c_0 or l^1, then $(B(X), \omega)$ is not Polish.

Ghoussoub and Maurey showed the following characterization: A separable Banach space X has (PC) and its dual X^* is separable if and only if $X^{**}\setminus X$ is the countable union of weak* compact sets (K_n) such that $d(K_n, X) > 0$ [9].

THEOREM 1. *Suppose that X is a non-reflexive space with a separable dual and* (PC). *Then there exists a boundedly complete basic sequence $(x_n) \subset X$ so that $(x_1, x_2 - x_1, x_n - x_{n-1}, \ldots)$ is a shrinking basic sequence.*

PROOF. Let (K_n) be weak*-compact sets with $X^{**}\setminus X = \bigcup_n K_n$ and $d(K_n, X) \geqq \delta_n > 0$. Let ε be any positive number and $x_1^{**} \in X^{**}\setminus X$, $\| x_1^{**} \| = 1$.

Let (g_n) be a dense sequence in X and let (d_n) be a dense sequence in X^*. Choose $(\varepsilon_n)_{n\geqq 0}$ so that $0 < \varepsilon_n < 1$ for all $n \geqq 0$ and so that $\Pi_{i=1}^{\infty} (1 - \varepsilon_i) \geqq 1 - \varepsilon_0$. There is an $f_1 \in S(X^*)$ such that $x_1^{**}(f_1) \geqq 1 - \varepsilon_1$. Let F_1 denote the 1-dimensional subspace of X^{**} spanned by x_1^{**}. By the local reflexivity principle [14], there is a one-to-one operator $T_1 : F_1 \overset{1-1}{\to} X$ so that

$$\text{(I)} \qquad \begin{cases} (1) \ \| T_1 \| \ \| T_1^{-1} \| \leqq 1 + \varepsilon_0, \\ (2) \ f_1(T_1 x_1^{**}) = x_1^{**}(f_1), \\ (3) \ d_1(T_1 x_1^{**}) = x_1^{**}(d_1), \\ \text{We put } x_1 = T_1 x_1^{**}. \end{cases}$$

Since $B([x_1])$ is compact and $[K_1 + \delta_1/2B(X^{**})] \cap B([x_1]) = \varnothing$, there exists a finite covering $\{V_j, j \in E_1\}$ for $B([x_1])$ such that V_j is a $\sigma(X^{**}, X^*)$ elementary open set disjoint of $K_1 + \delta_1/2B(X^{**})$ for each j. Let N_2 be the finite set of functionals that determines $\{V_j, j \in E_1\}$. Let M_2 be a 2^{-1}-norming set for $[x_1, g_1]$. Let F_2 denote the two-dimensional subspaces of X^{**} spanned by x_1^{**} and $x_1^{**} - x_1$. There is a set $Z_2 = \{e_1, \ldots, e_{N(2)}\} \subset S(F_2)$ which forms a $\varepsilon_2/2$ net for $S(F_2)$. Pick $e_1^*, \ldots, e_{N(2)}^* \in S(X^*)$ so that $e_i^*(e_i) > 1 - \varepsilon_2/2$. And put $Z_2^* = \{e_1^*, \ldots, e_{N(2)}^*\}$.

Let $Y_2 = N_2 \cup M_2 \cup Z_2^* \cup \{d_1, d_2\}$. By the local reflexivity principle, there is a one-to-one operator $T_2 : F_2 \overset{1-1}{\to} X$ such that

(1) $\| T_2 \| \ \| T_2^{-1} \| \leqq 1 + \varepsilon_0$,

(2) $f_1(T_2 x^{**}) = x^{**}(f_1)$, $\forall x^{**} \in F_2$,

(3) $e^*(T_2 x_1^{**}) = x_1^{**}(e^*)$, $\forall e^* \in Y_2$,

(4) $T_2 x_1 = x_1$.

We put $T_2 x_1^{**} = x_2$.

We repeat the above procedure. Inductively, we find for all n a finite covering $\{V_j, j \in E_{n-1}\}$ for $B([x_1, x_2, \ldots, x_{n-1}])$ such that V_j is a $\sigma(X^{**}, X^*)$ elementary open set disjoint of $\bigcup_{i \leqq n-1} (K_i + \delta_i/2B(X^{**}))$ for each j. N_n is the finite set of functionals that determines $\{V_j, j \in E_{n-1}\}$, M_n is a 2^{-n}-norming set for $[x_1, \ldots, x_{n-1}, g_1, \ldots, g_{n-1}]$, $F_n = [x_1^{**}, x_1^{**} - x_1, \ldots, x_1^{**} - x_{n-1}]$, $Z_n = \{e_1, \ldots, e_{N(n)}\}$ a $\varepsilon_n/2$-net for $S(F_n)$, $Z_n^* = \{e_1^*, \ldots, e_{N(n)}^*\} \subset S(X^*)$ such that $e_i^*(e_i) > 1 - \varepsilon_n/2$.

$$Y_n = \left(\bigcup_{i<n} Y_i \right) \cup N_n \cup M_n \cup Z_n^* \cup \{d_n\}.$$

There exists an operator $T_n : F_n \xrightarrow{1-1} X$ such that

(1) $\| T_n \| \, \| T_n^{-1} \| \leqq 1 + \varepsilon_0$,

(2) $f_1(T_n x^{**}) = x^{**}(f_1)$, $\forall x^{**} \in F_n$,

(3) $e^*(T_n x_1^{**}) = x_1^{**}(e^*)$, $\forall e^* \in Y_n$,

(4) $T_n e = e$, $\forall e \in F_n \cap X$.

And we put $T_n x_1^{**} = x_n$.

In [7], we proved that under such conditions (x_n) is a non-shrinking basic sequence and $(x_n - x_{n-1})_{n \geqq 1}$ is a basic sequence. Moreover, consider $f \in X^*$; then there is d_i such that $\| f - d_i \| \leqq \varepsilon$, and for $n > i$, we get

$$|f(x_n - x_1^{**})| = |(f - d_i)(x_n - x_1^{**})|,$$

consequently $x_n \xrightarrow{\omega^*} x_1^{**}$. And the sequence $(x_n - x_{n-1})$ is non-boundedly complete.

We prove now that (x_n) is boundedly complete. Indeed, let (a_n) be a sequence of scalars such that $\sup_n \| \Sigma_{i=1}^n a_i x_i \| < \infty$. Put $s_n = \Sigma_{i=1}^n a_i x_i$. (s_n) contains a weak* convergent subsequence $(s_{n_k})_k$. Let s be the weak* limit of (s_{n_k}) in X^{**}. We have for $k > p$,

$$\left| f_1 \left(\sum_{i=n_p+1}^{n_k} a_i x_i \right) \right| = \left| \sum_{i=n_p+1}^{n_k} a_i \right| \, |f_1(x_1^{**})| \xrightarrow[k,p \to \infty]{} 0.$$

Thus $(\Sigma_{i=1}^{n_k} a_i)_k$ is convergent. Then there exists a subsequence $(m_j)_j \subset (n_k)_k$ such that for each j,

$$\left| \sum_{i=m_j+1}^{m_{j+1}} a_i \right| \leqq 2^{-j-1} \delta_{n_j}.$$

If $s \notin X$, there exists p such that $s \in K_{n_p}$. We can assume without loss of generality that $s_{n_l} \in B(X)$ for each $l > p$. There is a weak* closed neighborhood of s_{m_p} ($s_{m_p} \in E_{m_p}$) of the form $W = \{y^{**}, \sup_\alpha |\langle y^{**} - s_{m_p}, y_\alpha \rangle| \leqq \varepsilon\}$ such that $(y_\alpha) \subset Y_{m_p+1}$ and

$$(*) \qquad W \cap \left[\bigcup_{i \leqq m_p} (K_i + \delta_i/2B(X^{**})) \right] = \varnothing .$$

For $n \geqq m_p$, we have

$$\left\langle s_n - \sum_{i=m_p+1}^{n} a_i x_1^{**} - s_{m_p}, y_\alpha \right\rangle = \left\langle \sum_{i=m_p+1}^{n} a_i x_i - \sum_{i=m_p+1}^{n} a_i x_1^{**}, y_\alpha \right\rangle$$
$$= 0.$$

Hence $s_n - \sum_{i=m_p+1}^{n} a_i x_1^{**} \in W$. And $s - \sum_{i=m_p+1}^{\infty} a_i x_1^{**} \in W$. But

$$\left\| \sum_{i=m_p+1}^{\infty} a_i x_1^{**} \right\| = \left| \sum_{i=m_p+1}^{\infty} a_i \right| \leqq 2^{-1} \delta_{n_p}.$$

And $s - \sum_{i=m_p+1}^{\infty} a_i x_1^{**} \in (K_{n_p} + 2^{-1}\delta_{n_p} B(X^{**})) \cap W$, which is a contradiction to $(*)$ since $m_p \geqq n_p$.

It follows that $s \in S$; hence $\forall \varepsilon > 0$, there exists p such that

$$\| g_{n_p} - s \| \leqq \varepsilon \quad \text{and} \quad \left| \sum_{n_p+1}^{n_l} a_i \right| \leqq \varepsilon \quad \text{for all } l < p.$$

There exists y in Y_{n_p+1}, $\| y \| \leqq 1$ and

$$\langle y, s_{n_p} - g_{n_p} \rangle \geqq \| s_{n_p} - g_{n_p} \| (1 + 2^{-(n_p+1)})^{-1}.$$

Let K be such that for all $k > K$, $|\langle y, s - s_{n_k} \rangle| \leqq \varepsilon$. Note that for $k > p$, we have

$$\left\langle y, s_{n_k} - g_{n_p} - \sum_{i=n_p+1}^{n_k} a_i x_1^{**} \right\rangle = \langle y, s_{n_p} - g_{n_p} \rangle,$$

and for $k > \max(K, p)$,

$$
\begin{aligned}
|\langle y, s-g_{n_p}\rangle| &\geqq \left|\left\langle y, s_{n_k}-g_{n_p}-\sum_{i=n_p+1}^{n_k} a_i x_i^{**}\right\rangle\right| - |\langle y, s-s_{n_k}\rangle| \\
&\quad - \left|\left\langle y, \sum_{i=n_p+1}^{n_k} a_i x_i^{**}\right\rangle\right| \\
&\geqq |\langle y, s_{n_p}-g_{n_p}\rangle| - 2\varepsilon.
\end{aligned}
$$

In addition

$$
\begin{aligned}
\| s_{n_p}-s \| &\leqq \| s_{n_p}-g_{n_p} \| + \| g_{n_p}-s \| \\
&\leqq \langle y, s_{n_p}-g_{n_p}\rangle(1+2^{-(n_p+1)})+\varepsilon \\
&\leqq (|\langle y, s-g_{n_p}\rangle| + 2\varepsilon)(1+2^{-(n_p+1)})+\varepsilon \\
&\leqq 3\varepsilon(1+2^{-(n_p+1)})+\varepsilon.
\end{aligned}
$$

It follows that (s_{n_k}) norm converges to s. And (s_n) norm converges to s because (x_n) is a basic sequence.

This proves that (x_n) is boundedly complete.

Suppose now $f \in [(x_n - x_{n-1})_{n \geq 1}]^*$ and let $\tilde{f}$ be a Hahn–Banach extension of f to an element in X^*. Then there is a subsequence (d_{m_i}) of (d_i) such that $\| d_{m_i} - \tilde{f} \| \to 0$. Now

$$d_{m_i}/[(x_j - x_{j-1})_{j=m_i+1}^{\infty}] = 0.$$

Hence

$$\lim_{i \to \infty} \| \tilde{f}/[(x_j - x_{j-1})_{j=m_i+1}^{\infty}] \| = 0$$

whence also $\| f/[(x_j - x_{j-1})_{j=i}^{\infty}] \| \to 0$. Thus $(x_n - x_{n-1})_{n \geq 1}$ is a shrinking basic sequence. ■

4. Applications

COROLLARY 2. *Every Banach space with* (PCA) *is somewhat quasireflexive.*

COROLLARY 3. [6] *If X has property* (PCA) *then X is somewhat reflexive.*

COROLLARY 4. [1] *If X^{**}/X is separable then X is somewhat quasireflexive.*

Recall that a Banach space X has the *Radon–Nikodym property* (RNP) if every weakly closed bounded subset of X has a denting point. Any space with (RNP) has property (PC).

COROLLARY 5. *If X has* (RNP) *and X^* is separable then X is somewhat quasireflexive.*

REMARK. It is possible to prove this corollary using weak* basic sequences [13] and the result of Ghoussoub and Maurey [9]: A separable Banach space X has (RNP) if and only if it has the asymptotic norming property.

A separable Banach space X is said to have the *asymptotic norming property* if there exists a separable Banach space Y such that X is isomorphic to a subspace of Y^* which verifies the following property:

$$\text{(ANP)}\qquad \begin{gathered} \text{if } (x_n) \subseteq X,\ x_n \xrightarrow{\omega^*} y^* \text{ and } \| x_n \| \to \| y^* \| \\ \text{then } \lim_n \| x_n - y^* \| = 0. \end{gathered}$$

The idea of the proof is then the following: Let $\| \cdot \|$ be a norm satisfying (ANP) and $x_1^{**} \in S(X^{**} \setminus X)$. We will use the same notations as in Theorem 1 and we will construct a basic sequence (x_n) such that the sequence $(x_1^{**}, x_1^{**} - x_1, x_1^{**} - x_2, \ldots)$ is weak* basic. At step 1, we have the relations (I) (see Theorem 1).

Put $Y_1 = \{d_1\}$.

At step n, there exists a finite subset G_n of $S(X^*)$ such that for every $f \in S(F_n^*)$ there exists $x^* \in G_n$ such that

$$| f(x^{**}) - x^{**}(x^*) | \leqq \varepsilon \| x^{**} \|, \quad \text{for every } x^{**} \in F_n.$$

Then then $Y_n = (\bigcup_{i<n} Y_i) \cup G_n \cup \{d_n\} \cup Z_n^*$. $(x_1^{**}, x_1^{**} - x_1, \ldots)$ and (x_n) are basic sequences [7]. Using the same ideas as in [13] it can be shown that $(x_1^{**}, x_1^{**} - x_1, \ldots)$ is weak* basic and if $s_n = \Sigma_{i=1}^n a_i x_i$ weak* converges to s then $\lim_n \| s_n \| = \| s \|$, and by (ANP), $\lim_n \| s_n - s \| = 0$.

It follows that (x_n) is a boundedly complete basic sequence; $(x_n - x_{n-1})$ is a basic sequence [7] which is shrinking (same proof as in Theorem 1).

Note added in proof. We would like to inform the reader that after this paper was accepted for publication we learned that S. Bellenot obtained independently our main result (S. Bellenot, *More quasireflexive spaces*, Proc. Am. Math. Soc., to appear).

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